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Another Characterization of Haar Subspaces

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1. Introduction

Let $C[a, b]$ denote the space of real valued continuous functions, f, defined on the compact nondegenerate real interval $[a, b]$. For $f \in C[a, b]$ the norm of f is defined by $\|f\| = \max_{a \leqslant x \leqslant b} |f(x)|$. Let W denote a finite dimensional subspace of $C[a, b]$. The function $\pi \in W$ is a best approximate to $f \in C[a, b]$ from W if

$$\|f - \pi\| \leqslant \|f - w\|,$$

for all $w \in W$. If the inequality is strict for all $w \in W$, $w \neq \pi$, then π is a unique best approximate to f from W. Further, if for $f \in C[a, b]$ there exist $\pi \in W$ and a positive number r, depending only on f, such that

$$\|f - w\| \geqslant \|f - \pi\| + r\|\pi - w\|,$$

for all $w \in W$ then π is said to be a strongly unique best approximate to f from W. An n-dimensional subspace W of $C[a, b]$ is called a Haar subspace if no nontrivial $w \in W$ vanishes at more than $n - 1$ distinct points of $[a, b]$.

In 1907 J. W. Young [5] proved that if W is a Haar subspace then every element of $C[a, b]$ possesses at most one best approximate from W. In 1918 A. Haar [2] proved that if every element of $C[a, b]$ possesses a unique best approximate from a finite dimensional subspace W then W is a Haar subspace. Thus a necessary and sufficient condition that every element of $C[a, b]$ possesses a unique best approximate from a finite dimensional subspace W is that W be a Haar subspace. (It is known that every element of $C[a, b]$ possesses at least one best approximate from W.)

In 1963 D. J. Newman and H. S. Shapiro [4] proved that every element of $C[a, b]$ possesses a strongly unique best approximate from a Haar subspace of $C[a, b]$. Since a strongly unique best approximate is also a unique best approximate it follows that a necessary and sufficient condition that every element of $C[a, b]$ possesses a strongly unique best approximate from a finite

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dimensional subspace W is that W be a Haar subspace. Thus if every element of $C[a, b]$ has a unique best approximate from a finite dimensional subspace W then in fact every element of $C[a, b]$ has a strongly unique best approximate from W.

If there exists more than one best approximate to $f \in C[a, b]$ from a subspace W then no one of these best approximates can be a strongly unique best approximate. A natural question is the following: for what subspaces W is it true that every element of $C[a, b]$ which possesses a unique best approximate from W also possesses a strongly unique best approximate from W? The purpose of this note is to characterize such subspaces.

2. Main Theorem

THEOREM. *Every element of $C[a, b]$ which possesses a unique best approximate from a finite dimensional subspace W also possesses a strongly unique best approximate from W if and only if W is a Haar subspace.*

The necessity part of this theorem is a corollary of the more technical Theorem 1 below. The sufficiency follows from the theorem of Newman and Shapiro referred to earlier.

The following lemma which is essential to the proof of Theorem 1 is given in [1].

LEMMA 1 (Generalized Kolmogorov Criterion). *Let $f \in C[a, b]$, W be a subspace of $C[a, b]$, and $\pi \in W$. The real number $r \geq 0$ satisfies*

$$\|f - w\| \geq \|f - \pi\| + r \|\pi - w\|,$$

for all $w \in W$ if and only if

$$\max_{x \in A} [f(x) - \pi(x)]\, w(x) \geq r \|f - \pi\| \|w\|,$$

for all $w \in W$, where

$$A = \{x \in [a, b] : |f(x) - \pi(x)| = \|f - \pi\|\}.$$

The following definitions are needed. For $g(x) \in C[a, b]$,

$$Z_g = \{x \in [a, b] : g(x) = 0\}$$

$$\operatorname{sgn} g(x) = \begin{cases} 1, & g(x) > 0, \\ 0, & g(x) = 0, \\ -1, & g(x) < 0. \end{cases}$$

For $w_1, \ldots, w_n \in C[a, b]$,

$$\langle w_1, \ldots, w_n \rangle = \{ g(x) \in C[a, b]:$$
$$g(x) = \sum_{i=1}^{n} a_i w_i(x),\ a_1, \ldots, a_n \text{ real constants}\}.$$

For $f \in C[a, b]$,

$$A(f) = A = \{x \in [a, b]: |f(x)| = \|f\|\},$$
$$Tf = \{\pi \in W: \|f - \pi\| \leqslant \|f - w\| \text{ for all } w \in W\}.$$

We say the function $f(x)$ defined on a subset P of $[a, b]$ can be extended continuously to $[a, b]$ if there exists $\bar{f}(x) \in C[a, b]$ such that $f(x) = \bar{f}(x)$ for $x \in P$. For convenience, we call the continuous extension $f(x)$.

The proof of Lemma 2 follows from elementary arguments which we omit.

LEMMA 2. *Assume the real-valued function $f(x)$ has been defined continuously on a set $G \subset [a, b]$ such that G is the union of a finite number of closed connected sets of $[a, b]$. Assume also that $|f(x)| \leqslant 1$ on G. Then f can be extended continuously to all of $[a, b]$ in such a way that $|f(x)| < 1$ on $[a, b] - G$.*

The proof of Theorem 1 is by induction. The proof for $n = 1$ is given in Lemma 3.

LEMMA 3. *Let $0 \neq w(x) \in C[a, b]$. Assume the subspace $W = \langle w \rangle$ is not a Haar subspace on $[a, b]$. Further, assume that the real-valued function $f(x)$ has been defined on a finite number of points of $[a, b]$, all contained in Z_w, such that $|f(x)| = 1$ for every x at which f is defined. Then f can be extended continuously to $[a, b]$ in such a way that the unique best approximate to f from W is not a strongly unique best approximate.*

Proof. Without loss of generality we assume $\|w\| = 1$. Since W is not a Haar subspace on $[a, b]$, Z_w, the zero set of $w(x)$ is not empty. Choose $x_0 \in Z_w$ such that $w(x) \neq 0$, $x \neq x_0$, x in some sufficiently small closed connected nondegenerate half-neighborhood of x_0, contained in $[a, b]$. Call this neighborhood $N(x_0)$. Without loss of generality, we assume $N(x_0)$ was chosen small enough such that there exists $x' \in [a, b] - N(x_0)$ such that $w(x') \neq 0$. It may be that $f(x_0)$ has been previously defined to be $+1$ or -1 by the hypothesis of the lemma; if not, define $f(x_0) = +1$. Without loss of generality, assume $\operatorname{sgn} w(x) = \operatorname{sgn} f(x_0) = f(x_0)$ in $N(x_0) - \{x_0\}$. We construct $f(x)$ on $\{x'\} \cup N(x_0)$ as follows:

$$f(x) = f(x_0)[1 - w^2(x)], \qquad x \in N(x_0),$$
$$f(x') = -\operatorname{sgn} w(x').$$

The function $f(x)$ is defined on $\{x'\} \cup N(x_0)$ and possibly on a finite number of points of Z_w, as in the hypothesis, so by Lemma 2, we extend $f(x)$ continuously to $[a, b]$ so that $|f(x)| < 1$ on $[a, b] - (\{x'\} \cup Z_w)$. Therefore, $\|f\| = 1$, $\{x_0, x'\} \subset A$, and $A \subset \{x'\} \cup Z_w$. Now we show that $Tf = \{0\}$.

Let a be a real number. Suppose $aw(x) \in Tf$. Then $\|f - aw(x)\| \leqslant 1$ since $\|f - 0\| = 1$. If $a > 0$, $|f(x') - aw(x')| = \big| 1 + a \,|w(x')|\big| > 1$. If $a < 0$, in $N(x_0) - \{x_0\}$ we have $|f(x) - aw(x)| = |f(x_0)[1 - w^2(x)] - aw(x)| = \big| 1 - w^2(x) - a\,|w(x)|\big| > 1$ for $0 < |w(x)| < -a$. Hence $a = 0$ and 0 is the unique best approximate to f from W.

To see that 0 is not a strongly unique best approximate, we check the generalized Kolmogorov criterion. We have

$$\max_{x \in A} (f(x) - 0)\, w(x) = \max\{0, f(x')\, w(x')\}$$
$$= \max\{0, -\operatorname{sgn} w(x') \cdot w(x')\} = 0.$$

Therefore the generalized Kolmogorov criterion,

$$\max_{x \in A} (f(x) - 0)\, w(x) \geqslant r\, \|f\|\, \|w\| \text{ for every } w \in W,$$

for some $r > 0$, fails to hold.

THEOREM 1. *Let $w_1(x), \ldots, w_n(x) \in C[a, b]$, be linearly independent functions. Assume the subspace $W = \langle w_1, \ldots, w_n\rangle$ is not a Haar subspace on $[a, b]$. Further, assume that the real-valued function $f(x)$ has been defined on a finite number of points of $[a, b]$, all contained in $\bigcap_{i=1}^n Z_{w_i}$, such that $|f(x)| = 1$ for each x for which f is defined. Then f can be extended continuously to $[a, b]$ in such a way that the unique best approximate to f from W is not a strongly unique best approximate.*

Proof. The theorem is proved for $n = 1$ in Lemma 3. Here we assume $n \geqslant 2$. We assume the theorem has been proved for $k = 1, 2, \ldots, n - 1$; i.e., if W is a k-dimensional subspace which is not a Haar subspace, $k \leqslant n - 1$, and $f(x)$ has been defined on a finite number of points of $[a, b]$ as in the statement of the theorem, then $f(x)$ can be extended continuously to $[a, b]$ such that the unique best approximate to f from W is not strongly unique. We then show that the theorem can be proved for the case where W is an n-dimensional subspace which is not a Haar subspace.

Without loss of generality, assume $\|w_i\| = 1$, $1 \leqslant i \leqslant n$. Further, without loss of generality, we assume $w_n(x) = 0$ on $\{x_1, \ldots, x_n\}$, n distinct points of $[a, b]$. Consider the subspace $W_1 = \langle w_1, \ldots, w_{n-1}\rangle$. If W_1 is not a Haar subspace on $\{x_1, \ldots, x_n\}$ then we can choose a basis for W_1, $w_1', \ldots, w_{n-1}'$

and a rearrangement of the points $\{x_1,..., x_n\}$, namely $\{x_1',..., x_n'\}$, such that $w_{n-1}'(x) = 0$ on $\{x_1',..., x_{n-1}'\}$. For convenience we drop the "′" notation from the functions $w_1',..., w_{n-2}'$, and from the points $x_1',..., x_n'$. (We keep the "′" on $w_{n-1}'(x)$.) Consider the subspace $W_2 = \langle w_1,..., w_{n-2}\rangle$. If W_2 is not a Haar subspace on $\{x_1,..., x_{n-1}\}$ then we can choose a basis for W_2, $w_1',..., w_{n-2}'$ and a rearrangement of the points $\{x_1,..., x_{n-1}\}$, namely $\{x_1',..., x_{n-1}'\}$, such that $w_{n-2}'(x) = 0$ on $\{x_1',..., x_{n-2}'\}$. For convenience we drop the "′" notation from the functions $w_1',..., w_{n-3}'$, and from the points $x_1',..., x_{n-1}'$. Continuing in this manner as long as possible, we define $w_n, w_{n-1}',..., w_{k+1}'$, where $0 \leqslant k \leqslant n-1$, inductively as above. When $k = n-1$ the set $w_n, w_{n-1}',..., w_{k+1}'$ reduces to the single function w_n. There are the following two cases to consider:

(I) The integer k is such that $1 \leqslant k \leqslant n-1$ and such that $\langle w_1,..., w_k\rangle$ is a Haar subspace on $\{x_1,..., x_{k+1}\}$.

(II) The integer k as defined above is 0. Hence for every j, $1 \leqslant j \leqslant n-1$, $\langle w_1,..., w_j\rangle$ is not a Haar subspace on $\{x_1,..., x_{j+1}\}$. In particular, we have $\langle w_1(x)\rangle$ is not a Haar subspace on $\{x_1, x_2\}$.

Case I. The integer k, $1 \leqslant k \leqslant n-1$, is such that $\langle w_1,..., w_k\rangle$ is a Haar subspace on $\{x_1,..., x_{k+1}\}$, and when $k < n-1$, we have $\langle w_1,..., w_k, w_{k+1}',..., w_{k+j}'\rangle$ is not a Haar subspace on $\{x_1,..., x_{k+j+1}\}$ for every j such that $1 \leqslant j \leqslant n-k-1$. Note that our subspace W has as its basis the functions $w_1,..., w_k, w_{k+1}',..., w_{n-1}', w_n$. We now drop the "′" notation from the functions $w_{k+1}',..., w_{n-1}'$. We note that the definition of k, $1 \leqslant k \leqslant n-1$, insures that all the functions $w_{k+1}(x),..., w_n(x)$ vanish on the set $\{x_1,..., x_{k+1}\}$.

Since $\langle w_1,..., w_k\rangle$ is a Haar subspace on $\{x_1,..., x_{k+1}\}$, it follows directly from the definition of a Haar subspace that we can interpolate at k points of the set $\{x_1,..., x_{k+1}\}$. Let $w_i'(x) \in \langle w_1,..., w_k\rangle$, $1 \leqslant i \leqslant k$ have the following values:

$$\begin{array}{lll} w_1'(x_{k+1}) = 1; & w_1'(x_i) = 0, & i \neq 1, k+1, \\ w_2'(x_{k+1}) = 1; & w_2'(x_i) = 0, & i \neq 2, k+1, \\ \vdots & & \\ w_k'(x_{k+1}) = 1; & w_k'(x_i) = 0, & i \neq k, k+1. \end{array}$$

We note that $w_i'(x_i)$ is unspecified, but we know that $w_i'(x_i) \neq 0$, $1 \leqslant i \leqslant k$, since if not, the Haar condition on $\langle w_1,..., w_k\rangle$ would be violated. Since $w_1',..., w_k'$ are linearly independent on $\{x_1,..., x_{k+1}\}$, they are linearly independent on $[a, b]$. Thus, without loss of generality, we assume $w_i'(x) = w_i(x)$, $1 \leqslant i \leqslant k$. We construct $f(x)$ on $\{x_1,..., x_{k+1}\} \subset \bigcap_{j=k+1}^{n} Z_{w_j}$, as follows:

$$f(x_i) = \operatorname{sgn} w_i(x_i), \qquad 1 \leqslant i \leqslant k,$$
$$f(x_{k+1}) = -1.$$

We remark that in this step $f(x)$ has been defined to be $+1$ or -1 only on a finite number of points of $\bigcap_{i=k+1}^{n} Z_{w_i}$. We construct f continuously on the remainder of $[a, b]$ such that $\|f\| = 1$, 0 is the unique best approximate to f from $W' = \langle w_{k+1}, \ldots, w_n \rangle$, an $n-k$ "dimensional subspace of $C[a, b]$, and such that 0 is not a strongly unique best approximate from W'. Since $1 \leqslant n-k \leqslant n-1$, this is possible by the induction hypothesis.

Let a_i, $1 \leqslant i \leqslant n$ be real numbers and suppose $\bar{w}(x) = \sum_{i=1}^{n} a_i w_i(x) \in Tf$. Then we know $\|f - \bar{w}\| \leqslant 1$, since $\|f - 0\| = 1$. If $a_i < 0$ for some $i = 1, \ldots, k$, then

$$|f(x_i) - \bar{w}(x_i)| = |\operatorname{sgn} w_i(x_i) - a_i w_i(x_i)| = \big|1 - a_i\, |w_i(x_i)|\big| > 1.$$

Hence we have $a_i \geqslant 0$, $1 \leqslant i \leqslant k$. But

$$|f(x_{k+1}) - \bar{w}(x_{k+1})| = \left|-1 - \sum_{i=1}^{k} a_i\right| = 1 + \sum_{i=1}^{k} a_i > 1 \text{ if } a_i > 0$$

for some $i = 1, \ldots, k$. Hence $a_i = 0$ for all $i = 1, \ldots, k$, and $\bar{w}(x)$ has the form $\bar{w}(x) = \sum_{i=k+1}^{n} a_i w_i(x)$. Hence we seek the best approximation to $f(x)$ from $W' = \langle w_{k+1}, \ldots, w_n \rangle$. But by the way f was constructed, this is 0; hence $a_i = 0$, $k+1 \leqslant i \leqslant n$, and the proof for Case I is complete.

Case II. There is no integer k, $1 \leqslant k \leqslant n-1$ such that $\langle w_1, \ldots, w_k \rangle$ is a Haar subspace on $\{x_1, \ldots, x_{k+1}\}$. ($\langle w_1, \ldots, w_k \rangle$ is the subspace that results after $n-k$ steps in the constructive process described earlier and $\{x_1, \ldots, x_{k+1}\}$ is the corresponding set of points.) We have $\langle w_1 \rangle$ is not a Haar subspace on $\{x_1, x_2\}$, and all the functions $w_2'(x), \ldots, w_{n-1}'(x)$, $w_n(x)$ vanish on $\{x_1, x_2\}$. We drop the "′" notation from the functions $w_2', \ldots, w_{n-1}'$. Without loss of generality, assume $w_1(x_1) = 0$. Hence $x_1 \in \bigcap_{i=1}^{n} Z_{w_i}$.

Case A. We assume that for some $k \in \{1, \ldots, n\}$, the set

$$(Z_{w_1} \cap \cdots \cap Z_{w_{k-1}} \cap Z_{w_{k+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_k}$$

contains at least two distinct points of $[a, b]$. Denote these points by $\bar{x}_k$ and x_k'. We construct f on $\{\bar{x}_k, x_k'\} \subset \bigcap_{i=1, i \neq k}^{n} Z_{w_i}$ as follows:

$$f(\bar{x}_k) = \operatorname{sgn} w_k(\bar{x}_k),$$
$$f(x_k') = -\operatorname{sgn} w_k(x_k').$$

In this step, $f(x)$ has been defined to be $+1$ or -1 on a finite number of points of $\bigcap_{i=1, i \neq k}^{n} Z_{w_i}$. We construct f continuously on the remainder of

$[a, b]$ such that $\|f\| = 1$, 0 is the unique best approximate to f from the $n-1$ "dimensional subspace $W' = \langle w_1, ..., w_{k-1}, w_{k+1}, ..., w_n \rangle$, and such that 0 is not a strongly unique best approximate from W'. This is possible by the induction hypothesis.

Let a_i, $1 \leq i \leq n$, be real numbers. Suppose $\bar{w}(x) = \sum_{i=1}^{n} a_i w_i(x) \in Tf$. Then if $a_k < 0$, $|f(\bar{x}_k) - \bar{w}(\bar{x}_k)| = |\operatorname{sgn} w_k(\bar{x}_k) - a_k w_k(\bar{x}_k)| > 1$, while if $a_k > 0$, $|f(x_k') - \bar{w}(x_k')| > 1$. Hence $a_k = 0$, and $\bar{w}(x)$ has the form $\bar{w}(x) = \sum_{i=1}^{k-1} a_i w_i(x) + \sum_{i=k+1}^{n} a_i w_i(x)$. Hence we seek the best approximation to $f(x)$ from $W' = \langle w_1, ..., w_{k-1}, w_{k+1}, ..., w_n \rangle$. But by the way f was constructed, this is 0; hence $a_i = 0$, $1 \leq i \leq n$, and the proof for Case A is complete.

Case B. Assume the set $(Z_{w_1} \cap \cdots \cap Z_{w_{i-1}} \cap Z_{w_{i+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_i}$ does not contain at least two distinct points of $[a, b]$ for any $i = 1, ..., n$. We show that in this case, we can assume, without loss of generality, that $(Z_{w_1} \cap \cdots \cap Z_{w_{i-1}} \cap Z_{w_{i+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_i}$ contains exactly one point for every $i = 1, ..., n$. To see this, suppose $(Z_{w_2} \cap \cdots \cap Z_{w_n}) - Z_{w_1} = \varnothing$. There exists $x_1' \in [a, b]$ such that $w_1(x_1') \neq 0$. Let $I = \{i \neq 1 : w_i(x_1') \neq 0\}$. By our assumption, $I \neq \varnothing$. For each $i \in I$, let $\alpha_i \neq 0$, $\beta_i \neq 0$ be chosen such that $\alpha_i w_i(x_1') + \beta_i w_i(x_1') = 0$. Then we choose as our basis functions for W the following functions:

$$w_i(x), \qquad i \notin I,$$
$$\bar{w}_i(x) = \alpha_i w_i(x) + \beta_i w_i(x), \qquad i \in I.$$

This is possible since the functions $w_i(x)$, $i \notin I$ and $\bar{w}_i(x)$, $i \in I$ are all linearly independent. For convenience, we drop the "–" notation. Then we have $w_1(x_1') \neq 0$ and $w_i(x_1') = 0$ for $1 < i \leq n$. Now suppose we have that the set $(Z_{w_1} \cap \cdots \cap Z_{w_{i-1}} \cap Z_{w_{i+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_i}$ contains the point x_i' for every $i = 1, ..., k-1$, where $2 \leq k \leq n$. Suppose also that

$$(Z_{w_1} \cap \cdots \cap Z_{w_{k-1}} \cap Z_{w_{k+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_k} = \emptyset.$$

There exists $x_k' \in [a, b]$ such that $w_k(x_k') \neq 0$. Let $I = \{i \neq k : w_i(x_k') \neq 0\}$. By the assumption, $I \neq \varnothing$. For each $i \in I$, let $\alpha_i \neq 0$, $\beta_i \neq 0$ be chosen such that $\alpha_i w_k(x_k') + \beta_i w_i(x_k') = 0$. We choose as the basis functions for W the following functions:

$$w_i(x), \qquad i \notin I,$$
$$\bar{w}_i(x) = \alpha_i w_k(x) + \beta_i w_i(x), \qquad i \in I.$$

This is possible since the functions $w_i(x)$, $i \notin I$ and $\bar{w}_i(x)$, $i \in I$ are all linearly independent. Then we have $w_k(x_k') \neq 0$ and $\bar{w}_i(x_k') = 0$ for $i \in I$, $1 \leq i \leq n$, and $w_i(x_k') = 0$, $i \neq k$, $i \notin I$, $1 \leqslant i \leqslant n$. Also $\bar{w}_i(x_i') = \alpha_i w_k(x_i') + \beta_i w_i(x_i') = \beta_i w_i(x_i') \neq 0$ for $i \in I$, $1 \leqslant i \leqslant k-1$ and $w_i(x_i') \neq 0$ for $i \notin I$, $1 \leqslant i \leqslant k-1$. Further, $\bar{w}_i(x_j') = \alpha_i w_k(x_j') + \beta_i w_i(x_j') = 0$ for $i \in I$ and $w_i(x_j') = 0$ for $i \notin I$, for every $i \neq j$, $i, j = 1,..., k-1$. For convenience, we drop the "—" notation from $\bar{w}_i(x)$, $i \in I$. Hence we have

$$\{x_k'\} \subset (Z_{w_1} \cap \cdots \cap Z_{w_{k-1}} \cap Z_{w_{k+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_k}.$$

By the induction hypothesis, we conclude that

$$\{x_i'\} \subset (Z_{w_1} \cap \cdots \cap Z_{w_{i-1}} \cap Z_{w_{i+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_i}$$

holds for every $i = 1,..., n$. If one of the above sets contains more than one point, Case A applies. Hence, we have without loss of generality, that $(Z_{w_1} \cap \cdots \cap Z_{w_{i-1}} \cap Z_{w_{i+1}} \cap \cdots \cap Z_{w_n}) - Z_{w_i} = \{x_i'\}$ for every $i = 1,..., n$. Without loss of generality we assume $\| w_i \| = 1$ holds for every $1 \leqslant i \leqslant n$. We now claim that without loss of generality we can assume that for each $i = 1,..., n$, there exist at most $n - 1$ points of $[a, b]$ where $w_i(x) = 0$ and $w_j(x) \neq 0$, each $j \neq i$. Suppose for some i there exists a set of n point $\{\bar{x}_1,..., \bar{x}_n\}$ such that $w_i(x) = 0$ on $\{\bar{x}_1,..., \bar{x}_n\}$ and for some $j \neq i$, $w_j(x) \neq 0$ on $\{\bar{x}_1,..., \bar{x}_n\}$. We show that then Case I applies. Indeed, we have $w_i(x) = 0$ on $\{\bar{x}_1,..., \bar{x}_n\}$. We relabel the functions $w_1,..., w_n$ so that $w_j(x) = \bar{w}_1(x)$ and $w_i(x) = \bar{w}_n(x)$. The relabeling of the functions $w_k(x)$, $k \neq i, j$ as $\bar{w}_2,..., \bar{w}_{n-1}$ is arbitrary. Hence we have $W = \langle \bar{w}_1,..., \bar{w}_n \rangle$ where $\bar{w}_n(x) = 0$ on $\{\bar{x}_1,..., \bar{x}_n\}$ and $\bar{w}_1(x) \neq 0$ on $\{\bar{x}_1,..., \bar{x}_n\}$. We drop the "—" notation for convenience.

Consider the subspace $W_1 = \langle w_1,..., w_{n-1} \rangle$. If W_1 is not a Haar subspace on $\{x_1,..., x_n\}$ there exists $w'_{n-1} \in W_1$ such that $w'_{n-1}(x)$ vanishes on $\{x_1',..., x'_{n-1}\} \subset \{x_1,..., x_n\}$. Since $w_1(x)$ and $w'_{n-1}(x)$ are linearly independent on $[a, b]$, we extend them to a basis of $W_1 = \langle w_1, w_2',..., w'_{n-1} \rangle$. We drop the "′" notation from the functions and the points for convenience. Consider the subspace $W_2 = \langle w_1,..., w_{n-2} \rangle$. If W_2 is not a Haar subspace on $\{x_1,..., x_{n-1}\}$, there exists $w'_{n-2}(x) \in W_2$ and $\{x_1',..., x'_{n-2}\} \subset \{x_1,..., x_{n-1}\}$ such that $w'_{n-2}(x) = 0$ on $\{x_1',..., x'_{n-2}\}$. Since $w_1(x) \neq 0$ on $\{x_1',..., x'_{n-2}\}$, $w_1(x)$ and $w'_{n-2}(x)$ are linearly independent on $[a, b]$, so we can extend them to a basis for W_2. Let the functions $w_1, w_2',..., w'_{n-2}$ form the basis. For convenience we drop the "′" notation.

We continue to define $w_n, w_{n-1},..., w_k$ where $2 \leqslant k \leqslant n-1$ inductively as above as long as possible. Since $w_1(x) \neq 0$ on $\{x_1,..., x_n\}$, we know that if in the process described above there exists no $k > 1$ such that $\langle w_1,..., w_k \rangle$ is a Haar subspace on $\{x_1,..., x_{k+1}\}$ it is true for $k = 1$, i.e., $\langle w_1 \rangle$ is a Haar subspace on $\{x_1, x_2\}$, and hence Case I applies. Thus in Case II we have the following two properties:

(1) $(\bigcap_{i=1, i \neq k}^{n} Z_{w_i}) - Z_{w_k} = \{x_k'\}$ for every k, $1 \leqslant k \leqslant n$.

(2) For each i, there exist at most $n - 1$ points where $w_i(x) = 0$ and $w_j(x) \neq 0$ each $j \neq i$, $1 \leqslant i, j \leqslant n$.

(We note that Case II is characterized by the fact that $\bigcap_{i=1}^{n} Z_{w_i} \neq \varnothing$.)

Choose any point where $w_1(x)$ does not vanish. For explicitness, we choose x_1'. Let x_{n+1} be a closest point to x_1' from $\bigcap_{i=1}^{n} Z_{w_i}$. The point x_{n+1} exists since $\bigcap_{i=1}^{n} Z_{w_i}$ is a closed, nonempty set.

We show that there exists some closed connected nondegenerate half-neighborhood of x_{n+1}, $N_1(x_{n+1})$, contained in $[a, b]$ such that $w_1(x) \neq 0$ in $N_1(x_{n+1}) - \{x_{n+1}\}$. If not, then in the interval between x_1' and x_{n+1}, $w_1(x)$ must have an infinite number of distinct zeros; call them $\{z_i\}_{i=1}^{\infty}$. Then the functions $w_2, ..., w_n$ are each nonzero on at most $n - 1$ points of $\{z_i\}_{i=1}^{\infty}$. Hence all the functions $w_2, ..., w_n$ are zero on all but at most $(n - 1)^2$ points of $\{z_i\}_{i=1}^{\infty}$. But then x_{n+1} is not the closest point of $\bigcap_{i=1}^{n} Z_{w_i}$ to x_1'. Hence $w_1(x)$ has a finite number of zeros between x_1' and x_{n+1}, and for some sufficiently small nondegenerate half-neighborhood of x_{n+1}, $N_1(x_{n+1})$, contained in $[a, b]$, $w_1(x) \neq 0$ in $N_1(x_{n+1}) - \{x_{n+1}\}$. Since the function $w_1(x) \neq 0$ in $N_1(x_{n+1})$, each of the functions $w_2, ..., w_n$ has at most $n - 1$ zeros in $N_1(x_{n+1})$ because of property (2) above of the subspace W. Hence there exists a sufficiently small connected closed nondegenerate half-neighborhood of x_{n+1}, $N(x_{n+1})$, contained in $[a, b]$, such that $w_i(x) \neq 0$ for $x \in N(x_{n+1}) - \{x_{n+1}\}$ holds for every $i = 1, ..., n$.

It may be that $f(x_{n+1})$ has been previously defined to be $+1$ or -1; this is possible by the hypothesis of the theorem, since $x_{n+1} \in \bigcap_{i=1}^{n} Z_{w_i}$. If not, define $f(x_{n+1})$ to be $+1$. Without loss of generality, assume sgn $w_i(x) = -f(x_{n+1})$ for $x \in N(x_{n+1}) - \{x_{n+1}\}$ holds for every $i = 1, ..., n$. We construct $f(x)$ on $\{x_1', ..., x_n'\} \cup N(x_{n+1})$ as follows:

$$f(x_i') = \operatorname{sgn} w_i(x_i'), \qquad i = 1, ..., n,$$

$$f(x) = f(x_{n+1})\left[1 - \prod_{i=1}^{n} |w_i(x)|\right], \qquad x \in N(x_{n+1}).$$

As in Lemma 2, we extend $f(x)$ continuously to $[a, b]$ such that $\|f\| = 1$, and $|f(x)| < 1$ except possibly for some of the points of $Z_{w_1} \cup \{x'_1\}$; all the extreme points of $f(x) - 0$ are contained in the set $Z_{w_1} \cup \{x_1'\}$.

Let a_i, $1 \leqslant i \leqslant n$, be real numbers. Suppose $\bar{w}(x) = \sum_{i=1}^{n} a_i w_i(x) \in Tf$. Then $\|f - \bar{w}\| \leqslant 1$, since $\|f - 0\| = 1$. If $a_i < 0$ for some $i = 1, ..., n$, then $|f(x_i') - \bar{w}(x_i')| = |\operatorname{sgn} w_i(x_i') - a_i w_i(x_i')| = \big|1 - a_i |w_i(x_i')|\big| > 1$. Hence,

$a_i \geqslant 0$ for every $i = 1,\ldots, n$. However, if $a_k > 0$ for some k, then in $N(x_{n+1}) - \{x_{n+1}\}$ we have

$$\begin{aligned} |f(x) - \bar{w}(x)| &= \left| f(x_{n+1})\left[1 - \prod_{i=1}^{n} | w_i(x)|\right] - \sum_{i=1}^{n} a_i w_i(x)\right| \\ &= \left| f(x_{n+1})\left[1 - \prod_{i=1}^{n} | w_i(x)|\right] + f(x_{n+1}) \sum_{i=1}^{n} a_i \mid w_i(x)|\right| \\ &= \left| 1 - \prod_{i=1}^{n} | w_i(x)| + \sum_{i=1}^{n} a_i \mid w_i(x)|\right| > 1, \end{aligned}$$

for $x \in N(x_{n+1})$ such that $\prod_{i=1, i\neq k}^{n} | w_i(x)| < a_k$. The existence of $x \in N(x_{n+1}) - \{x_{n+1}\}$ such that $\prod_{i=1, i\neq k}^{n} | w_i(x)| < a_k$ follows from the fact that $\lim_{x\to x_{n+1}} \prod_{i=1, i\neq k}^{n} | w_i(x)| = 0$. Thus $a_i = 0$ for every $i = 1,\ldots, n$ and 0 is the unique best approximate to $f(x)$ from W. To see that 0 is not a strongly unique best approximate, we check the generalized Kolmogorov criterion. We have

$$\begin{aligned} &\max_{x\in A} (f(x) - 0)(-w_1(x)) \\ &\quad = \max \{0, \operatorname{sgn} w_1(x_1')(-w_1(x_1'))\} = 0. \end{aligned}$$

Therefore the generalized Kolmogorov criterion,

$$\max_{x\in A} (f(x) - 0)\, w(x) \geqslant r \,\|f\|\, \|w\| \text{ for every } w \in W,$$

for some $r > 0$, fails to hold.

In a future note the authors will discuss the application of the result of this paper to the notion of generalized strong unicity [3].

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